Routing Games and The Price Of Anarchy

Tutorial 5 Solutions

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1 Problem 1

The game in matrix form

Player II

Player I	$2 \to 4$	$2 \to 3 \to 4$
$1 \to 4$	3, 4	3, 3
$1 \to 3 \to 4$	2, 4	3, 4

The potential of a strategy $\boldsymbol{p}$ is defined as

$$\Phi(\boldsymbol{p}) = \sum_{e \in \boldsymbol{E}} \sum_{i=1}^{n_e(\boldsymbol{p})} c_e(i) \tag{1}$$

where $n_e(\boldsymbol{p})$ represents the number of players that play edge e when the overall strategy of play is $\boldsymbol{p}$. So the 4 squares below iterate all possible strategies, and for each strategy, we add up the costs as per the definition above.

Player II

Player I	$2 \to 4$	$2 \to 3 \to 4$
$1 \to 4$	3+4 = 7	3 + (1 + 2) = 6
$1 \to 3 \to 4$	(1+1)+4 = 6	1 + 2 + (1+2) = 6

Notice that last (1+2) in the bottom right cell. That happens because two players use the edge $3 \to 4$, and thus we have to account for it when $c_{3\to4}(1) + c_{3\to4}(2) = 3$.

There are 3 pure Nash equilibria, those that have potential 6. The PoA is 7/6.

2 Problem 2

Upper Bound For this problem, we first re-derive the POA for linear cost functions (and fill in some missing details from the slides). Make sure you fully understand all the steps in this proof. If you do, then the rest follows with some algebra magic (shown at the end). The first piece of magic we need is the following mathematical fact.

Fact 1 For any $x, y \in \mathbb{N}$, we have $3(x+1)y \leq x^2 + 5y^2$.

Let $c_e(x) = \alpha_e x + \beta_e$ be the linear cost function for some non-negative α_e, β_e for each edge $e \in \boldsymbol{E}$, where $\boldsymbol{E}$ represent all edges in the network. Let $\boldsymbol{p} = p_1, \ldots, p_n$ denote the player actions in the NE, where p_j is the path from source to target chosen by player $j \in N$. Similarly, let $\boldsymbol{p}^\star = p_1^\star, \ldots, p_n^\star$ be the optimal strategy which minimises overall cost. Let $n_e(\boldsymbol{p})$ denote the number of players that take edge e when we are playing with strategy $\boldsymbol{p}$. Thus, the total cost of playing strategy $\boldsymbol{p}$ is

$$\begin{aligned}
\sum_{i \in N} C_i(\boldsymbol{p}) &= \sum_{i \in N} \sum_{e \in p_i} c_e(n_e(\boldsymbol{p})) && (2) \\
&= \sum_{e \in \boldsymbol{E}} c_e(n_e(\boldsymbol{p})) \cdot n_e(\boldsymbol{p}) && (3) \\
& && (4)
\end{aligned}$$

Now the cost for player $i \in N$, when everyone plays the NE, is the cost of the edges it traverses in is path p_i.

$$\begin{aligned}
C_i(\boldsymbol{p}) &= \sum_{e \in p_i} c_e(n_e(\boldsymbol{p})) && (5) \\
&\leq \sum_{e \in p_i^\star} c_e(n_e(\boldsymbol{p_{-i}}; p_i^\star)) && (6) \\
&\leq \sum_{e \in p_i^\star} c_e(n_e(\boldsymbol{p}) + 1) && (7)
\end{aligned}$$

(6): Player i changes from from p_i to $p_i^\star$, but as $\boldsymbol{p}$ is NE, it cannot improve it's cost. (7): By swiching to $p_i^\star$ player i can add at most 1 to every edge. Combining (2) with (7), we get

$$\begin{aligned}
\sum_{i \in N} C_i(\boldsymbol{p}) &\leq \sum_{i \in N} \sum_{e \in p_i^\star} c_e(n_e(\boldsymbol{p}) + 1) && (8) \\
&= \sum_{e \in \boldsymbol{E}} c_e(n_e(\boldsymbol{p}) + 1) \cdot n_e(\boldsymbol{p}^\star) && (9) \\
&\leq \sum_{e \in \boldsymbol{E}} \frac{1}{3} c_e(n_e(\boldsymbol{p}))\, n_e(\boldsymbol{p}) + \frac{5}{3} c_e(n_e(\boldsymbol{p}^\star))\, n_e(\boldsymbol{p}^\star) && (10) \\
&= \frac{1}{3} \sum_{e \in \boldsymbol{E}} c_e(n_e(\boldsymbol{p}))\, n_e(\boldsymbol{p}) + \frac{5}{3} \sum_{e \in \boldsymbol{E}} c_e(n_e(\boldsymbol{p}^\star))\, n_e(\boldsymbol{p}^\star) && (11) \\
&\leq \frac{1}{3} \sum_{i \in N} C_i(\boldsymbol{p}) + \frac{5}{3} \sum_{i \in N} C_i(\boldsymbol{p}^\star) && (12)
\end{aligned}$$

Finally, we get

$$\frac{\sum_{i \in N} C_i(\boldsymbol{p})}{\sum_{i \in N} C_i(\boldsymbol{p}^\star)} \leq \frac{\frac{5}{3}}{1 - \frac{1}{3}} = \frac{5}{2}$$

(10) follows from the derivation shown below. Let $x = n_e(\boldsymbol{p})$ and $y = n_e(\boldsymbol{p}^\star)$.

$$\begin{aligned}
c_e\left(n_e(\boldsymbol{p})+1\right)\cdot n_e\left(\boldsymbol{p}^\star\right) &= c_e(x+1)\,y && (13)\\
&= (\alpha_e(x+1)+\beta_e)y && (14)\\
&= \alpha_e(x+1)y+\beta_e y && (15)\\
&\le \alpha_e\left[\frac{1}{3}x^2+\frac{5}{3}y^2\right]+\beta_e y && (16)\\
&= \left[\frac{1}{3}\alpha_e xx+\frac{5}{3}\alpha_e yy\right]+\beta_e y && (17)\\
&\le \frac{1}{3}\alpha_e(x+\frac{\beta_e}{\alpha_e})x+\frac{5}{3}\alpha_e yy+\beta_e y && (18)\\
&= \frac{1}{3}\alpha_e(x+\beta_e)x+\frac{5}{3}y(\alpha_e y+\beta_e) && (19)\\
&= \frac{1}{3}c_e(x)\,x+\frac{5}{3}c_e(y)\,y && (20)
\end{aligned}$$

(24) comes from Fact 1. In a sense, all the work for this proof happened in expanding (10), and the expansion really relies on Fact 1. So the whole game again for the new cost function $c_e(x) = a_e(x+1)+b_e$ will be to come with a mathematical lemma such that I can take a product of two terms and split it into the sums.

Fact 2 Let $g(x) = x+1$, then $g(x+1)y \le \frac{1}{4}g(x)\cdot x + \frac{5}{4}g(y)\cdot y$.

You can take Fact 2 and substitute Equation (12) with $\frac{1}{4}$ and $\frac{5}{4}$ instead of $\frac{1}{3}$ and $\frac{5}{3}$, respectively.

$$\begin{aligned}
c_e\left(n_e(\boldsymbol{p})+1\right)\cdot n_e\left(\boldsymbol{p}^\star\right) &= c_e(x+1)\,y && (21)\\
&= (\alpha_e(x+1+1)+\beta_e)y && (22)\\
&= \alpha_e g(x+1)y+\beta_e y && (23)\\
&\le \alpha_e\left[\frac{1}{4}xg(x)+\frac{5}{4}yg(y)\right]+\beta_e y && (24)\\
&= \left[\frac{1}{4}\alpha_e xg(x)+\frac{5}{4}\alpha_e yg(y)\right]+\beta_e y && (25)\\
&\le \frac{1}{4}\alpha_e(x+1+\frac{\beta_e}{\alpha_e})x+\frac{5}{4}\alpha_e y(y+1)+\frac{5}{4}\beta_e y && (26)\\
&= \frac{1}{4}x(\alpha_e(x+1)+\beta_e)+\frac{5}{4}y[\alpha_e(y+1)+\beta_e] && (27)\\
&= \frac{1}{4}c_e(x)\,x+\frac{5}{4}c_e(y)\,y && (28)
\end{aligned}$$

The POA

$$\frac{\sum_{i\in N} C_i(\boldsymbol{p})}{\sum_{i\in N} C_i(\boldsymbol{p}^\star)} \le \frac{\frac{5}{4}}{1-\frac{1}{4}} = \frac{5}{3}$$

Remark 1 **Verifying Fact 2 and Fact 1 is quite easy once I tell you what the constants are. Just start with small values for x and y and check it holds. As the RHS grows much faster than the LHS as you increase x and y, if the check holds for small values, it must hold for large values.**

For the **lower bound**, we just need one example of a cost function to match the upper bound. The figure below achieves this (This is the same game as the slides lower bound)

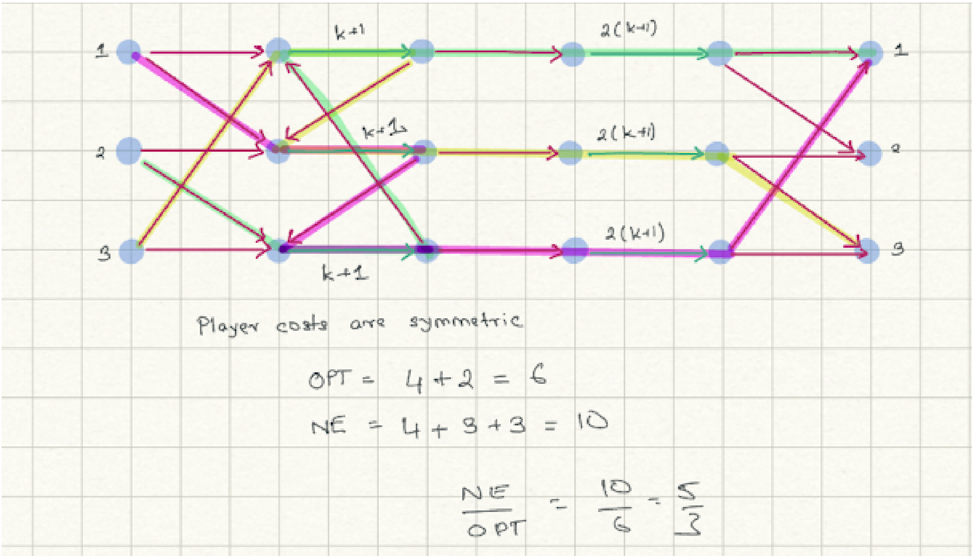

Figure 1: Change the right parallel edges to $c(k) = 2k$, in the original problem in the slides. The new cost function becomes $2(k+1)$ for the right paths and $k+1$ for the left paths. For OPT travel time per player, notice each payable edge is used once. So we get $2(1+1)+(1+1) = 6$, and for the NE per player, we have the player i uses player $i+1 \mod 3$ and $i+2 \mod 3$ left edges, and the right edges get used once per player. So $(2+1)+(2+1)+2(1+1) = 10$.

3 Problem 3

Want to show that.

$$L(x)y \leq \frac{1}{4}L(x)x + L(y)y \tag{29}$$

$$L(x) \leq \frac{x}{4y}L(x) + L(y) \tag{30}$$

$$L(x)(1 - \frac{x}{4y}) \leq L(y) \tag{31}$$

If $y = 0$, the whole thing is trivial, so assume without loss of generality $y \neq 0$. When $y \geq x$, as L is non-negative and non-decreasing, we have $L(x) \leq L(y)$, and therefore $yL(x) \leq yL(y) \leq yL(y) + \frac{1}{4}L(x)x$, as x, y are non-negative and so is $L(x)$. Similarly, if $y \leq \frac{x}{4}$, $L(x)y \leq \frac{x}{4}L(x) + yL(y)$, as y and $L(y)$ are non-negative.

This leaves us the case that $\frac{x}{4} \leq y \leq x$, we will use the fact L is concave. A function f is concave if for any u, v in the function's domain, and $\alpha \in [0,1]$

$$f(\alpha u + (1-\alpha)v) \geq \alpha f(u) + (1-\alpha)f(v)$$

Setting $L = f$, $u = 0$, $v = x$ and $\alpha = \frac{x}{4y} \in [0,1]$, we get

$$\frac{x}{4y}L(0) + (1 - \frac{x}{4y})L(x) \leq L((1 - \frac{x}{4y})x) \tag{32}$$
$$\frac{x}{4y}L(0) + (1 - \frac{x}{4y})L(x) \leq L(y) \tag{33}$$
$$L(x)(1 - \frac{x}{4y}) \leq L(y) \tag{34}$$

(33) comes from the fact that L is non-decreasing, and $(x - \frac{x^2}{4y}) \leq y$. In equation (34), as we know $y \neq 0$ and L and x are non-negative, we are only making the LHS smaller by removing a non-negative quantity.

To see why $(x - \frac{x^2}{4y}) \leq y$, observer

$$x(1 - \frac{x}{4y}) \leq y \tag{35}$$
$$x(4y - x) \leq 4y^2 \tag{36}$$
$$4y^2 + x^2 - 2 \cdot 2y \cdot x \geq 0 \tag{37}$$
$$(x - 2y)^2 \geq 0 \tag{38}$$

As $x, y \in [0, \infty)$, the square of $(x - 2y)$ must be non-negative. Once you prove this quantity, the proof that the POA is upper bound is identical to the material in the slides. Instead of using $xy \leq \frac{1}{4}x^2 + y^2$ for $x, y \in \mathbb{R}^+$, we substitute $c_e(\cdot)$, with L and use equation (34).

4 Problem 4

Player I \ Player II	$2 \to 3$	$2 \to 1 \to 3$
$1 \to 3$	4	$a+2$
$1 \to 2 \to 3$	$a+2$	$a+4$

So the optimal strategy is $1 \to 3, 2 \to 3$ if $a > 2$, and $1 \to 3, 2 \to 1 \to 3$ is optimal if $1 \leq a \leq 2$.

For $a > 1$, NE has social cost 4. For $a \leq 1$,the optimal solution is a NE.

$$PoS = \begin{cases} \frac{4}{2+a}, & \text{for } 1 < a < 2 \\ 1 & \text{otherwise} \end{cases}$$