A combinatorial Lemma : Pigeon Filtering

Let $m, n \in \mathbb{N}$ where (m will be the number of pigeons ; and L is the size of the refutation).
Also let $\alpha, w_0 \in [m]$ s.t $w_0 \geq \ln L$ and $w_0 \geq \alpha^2 \geq 4$. For any arbitrary set of L integer vectors of length m — $(\vec{x}(1), \dots, \vec{x}(L))$ where $\vec{x}(j) = (x_1(j), \dots x_m(j))$ for $j \in [L]$ and $x_k(j) \in \mathbb{N}$ $\forall k \in [m]$.

There exist a vector $\vec{r} = (r_1, \dots r_m)$ with $r_k \leq \left\lfloor \frac{\log m}{\log \alpha} \right\rfloor - 1$ $\forall k \in [m]$ such that for every $\ell \in [L]$ at least one of the following hold :

① $\left| \{ i \in [m] : x_i(\ell) \leq r_i \} \right| \geq w_0$

② $\left| \{ i \in [m] : x_i(\ell) \leq r_i + 1 \} \right| \leq O(\alpha w_0)$.

Let L, m, α, w_0 be as defined. Define $W(\vec{x}) = \sum_{i \in [m]} \alpha^{-x_i}$. Fix $\ell \in [L]$.

There exists constants γ, γ' st if

(i) $W(\vec{x}(\ell)) \geq \frac{\gamma' w_0}{\alpha} \Rightarrow$ condition 1

(ii) $W(\vec{x}(\ell)) \leq \frac{\gamma' w_0}{\alpha} \Rightarrow$ condition 2.

If we show the $\Rightarrow$ we are done as $W(\vec{x}(\ell))$ must be in one of the cases

Define a distribution $\mu \in \Delta([t])$ where $t = \left\lfloor \frac{\log m}{\log \alpha} \right\rfloor - 1$; where $\Pr_{x \leftarrow \$ \mu} \{ x = i \} = \beta \cdot \alpha^{-i}$

where $\beta = \frac{\alpha - 1}{1 - \alpha^{t}}$. μ is well defined as $\sum_{i \in [m]} \Pr_{x \leftarrow \$ \mu} \{x = i\} = 1$

(by the geometric distribution).

This is picked to normalize the dist.

Define $\vec{r} = (r_1, \dots r_m)$ where $r_i \leftarrow \$ \mu$ $\forall i \in [m]$. Fix $\ell \in [L]$.

① $\boxed{\text{suppose } W(\vec{x}(\ell)) \geq \frac{\gamma' w_0}{\alpha}}$.

Observe for all $i \in [m]$ s.t $x_i(\ell) > t$; we have

$$\sum_{i : x_i(\ell) > t} \alpha^{-x_i(\ell)} \leq \sum_{i : x_i(\ell) > t} \alpha^{-(t+1)} = m \cdot \alpha^{-(t+1)} < \alpha$$

since $t = \left\lfloor \frac{\log m}{\log \alpha} \right\rfloor - 1$ and $\alpha \geq 2$.

$$\sum_{i : x_i(\ell) \leq t} \alpha^{-x_i(\ell)} \geq \frac{\gamma' w_0}{\alpha} - \alpha$$

$$= \frac{\gamma' w_0 - \alpha^2}{\alpha} \geq \frac{\gamma' w_0 - w_0}{\alpha} = \frac{w_0}{\alpha} (\gamma' - 1)$$

$\because w_0 \geq \alpha^2$

Also $\Pr_{r_i \leftarrow \$ \mu}\left[r_i \ge x_i(\ell)\right] = \sum_{x = x_i(\ell)}^{\infty} \beta \cdot \alpha^{-x}$

$$\ge \beta \cdot \alpha^{-x_i(\ell)}$$

Now consider the set $P(\ell, \vec{r}) := \{ i \in [m] : x_i(\ell) \le t \ \wedge \ r_i \ge x_i(\ell) \}$

$$\mathbb{E}_{\vec{r} \leftarrow \$ \otimes_m \mu}\left[|P(\ell, \vec{r})| \right] = \sum_{i : x_i(\ell) \le t} \Pr\left[r_i \ge x_i(\ell) \right]$$

$$\ge \beta \sum_{i : x_i(\ell) \le t} \alpha^{-x_i(\ell)}$$

$$\ge \beta \cdot \frac{w_0}{\alpha} (\gamma' - 1)$$

$$\ge \frac{w_0}{2} (\gamma' - 1)$$

$\beta = \frac{\alpha - 1}{1 - \alpha^{-t}}$

$\frac{\beta}{\alpha} = \underbrace{\left(\frac{\alpha - 1}{\alpha} \right)}_{\ge \frac{1}{2}} \underbrace{\frac{1}{1 - \alpha^{-t}}}_{> 1}$

$\ge \frac{1}{2}$

Now for arbitrary $\ell \in [L]$.

$$\Pr_{\vec{r}}\left[\underbrace{|P(\ell, \vec{r})| \le w_0}_{\text{condition 1 fails}} \right]$$

$$= \Pr_{\vec{r}} \left\{ |P(\ell, \vec{r})| - \overbrace{\mathbb{E}\left[|P(\ell, \vec{r})| \right]}^{= P} \le w_0 - \underbrace{\mathbb{E}\left[|P(\ell, \vec{r})| \right]}_{= P} \right\}$$

$$= \Pr_{\vec{r}} \left\{ P - |P(\ell, \vec{r})| \ge P - w_0 \right\}$$

$$\le \exp\left\{ - \frac{(P - w_0)^2}{2P + (P - w_0)} \right\}$$ By chernoff bound.

$$\le \exp\left\{ - \frac{(P - 2w_0)}{2} \right\}$$

$$\le \exp\left\{ -\frac{1}{4} \left(w_0 (\gamma' - 1) - 2w_0 \right) \right\} = \exp\left\{ - \frac{w_0}{4} \left((\gamma' - 1) - 2 \right) \right\}$$

Set $\gamma' \ge 13$.

$$\le \exp\left\{ - 2w_0 \right\}$$

$$\le L^2 \text{ — for any } \ell \in [L]$$

What we got was that
For any arbitrary $\ell \in [L]$
condition 1 fails with probability
at most $1/L^2$ when conditioning on
$W(\vec{r}(\ell)) \ge \gamma' w_0 / 2$.

Now consider case 2: $W(\vec{r}(\ell)) \le \frac{\gamma' w_0}{\alpha}$

$$\Pr_{r_i \leftarrow \$ \mu}[r_i \ge r_i(\ell) - 1] = \sum_{j = r_i(\ell)-1}^{t} \Pr[r_i = j]$$

$$= \beta \sum_{j = r_i(\ell)-1}^{t} \alpha^{-j} = \beta \cdot \left(\frac{\alpha^{-r(\ell)+2} - \alpha^{-t}}{\alpha - 1} \right)$$

geometric series & $\alpha \ge 2$
start from $j=0$ for both & diff.

$$= \frac{1}{1-\alpha^{-t}} \left[\alpha^{-r_i(\ell)+2} - \alpha^{-t} \right] \cdot \frac{\alpha^{+t}}{\alpha^{+t}}$$

$$= \frac{\alpha^{t - r_i(\ell) + 2} - 1}{\alpha^t - 1}$$

$$\le \frac{\alpha^{t - r_i(\ell) + 2}}{\alpha^t / 2} \qquad \because \alpha \ge 2$$

$$= 2\alpha^{2 - r_i(\ell)}.$$

Now define $Q(\vec{r}, \ell) := \{ i \in [m] : r_i \ge r_i(\ell) - 1 \}$

$$q := E[|Q(\vec{r},\ell)|] = \sum_{i \in [m]} \Pr[r_i \ge r_i(\ell) - 1] \le \sum_{i \in [m]} 2\alpha^{2 - r_i(\ell)}$$

$$= 2\alpha^2 \sum_{i \in [m]} \alpha^{-r_i(\ell)} = 2\alpha^2 \cdot W(\vec{r}(\ell))$$

$$\le 2 \cdot \alpha \cdot \gamma' w_0 \quad \text{(by assumption of case 2).}$$

Now.

$$\Pr_{\vec{r}} \left\{ |Q(\vec{r},\ell)| \ge \gamma w_0 \alpha \right\} \le \Pr_{\vec{r}} \left\{ |Q(\vec{r},\ell)| - q \ge \gamma \alpha w_0 - \underbrace{2\alpha\gamma' w_0}_{\ge q.} \right\}$$

condition 2 fails

Chernoff version
Let S be sum of $[0,1]$ RVs (not necessarily independent)

$$\Pr[S - E[S] \ge \beta] \le \exp\left\{ - \frac{\beta^2}{2E[S] + \beta} \right\}$$

$$\le \exp\left\{ - \frac{(\gamma\alpha w_0 - 2\alpha\gamma' w_0)^2}{2q + (\gamma\alpha w_0 - 2\alpha\gamma' w_0)} \right\}$$

$$\le \exp\left\{ - \frac{(\gamma\alpha w_0 - 2\alpha\gamma' w_0)^2}{4\alpha\gamma' w_0 + (\gamma\alpha w_0 - 2\alpha\gamma' w_0)} \right\}$$

$$= \exp\left\{ - \frac{(\alpha w_0)^2 (\gamma - 2\gamma')^2}{4\alpha\gamma' w_0 + (\gamma - 2\gamma')\alpha w_0} \right\}$$

$$\overset{(*)}{\le} \exp\{ -\alpha w_0 \}$$

$(*)$ $\frac{\alpha w_0 (\gamma - 2\gamma')^2}{(\gamma + 2\gamma')} \ge \alpha w_0 \frac{(\gamma + 2\gamma')}{(\gamma + 2\gamma')}$

$w_0 \ge \log L$, $\alpha \ge 2$ $\longrightarrow$ $\le L^{-2}$

$\gamma \ge 5\gamma'$ does the job

set γ st $(\gamma - 2\gamma')^2 \ge (\gamma + 2\gamma')$

We have for any $\ell \in [L]$; Let $p = \Pr_{\vec{r}}\left\{ W(\vec{r}(\ell)) \geq \frac{r' w_0}{\alpha} \right\}$.

$$\Pr_{\vec{r}}\left[\overline{1} \;\middle|\; W(\vec{r}(\ell)) \geq \frac{r' w_0}{\alpha} \right] \leq \frac{1}{L^2}$$

$$\Pr_{\vec{r}}\left[\overline{2} \;\middle|\; W(\vec{r}(\ell)) \leq \frac{r' w_0}{\alpha} \right] \leq \frac{1}{L^2}.$$

This is what we have shown.

$$\Pr_{\vec{r}}[\underbrace{\overline{1} \wedge \overline{2}}_{X(\ell)}] = \Pr_{\vec{r}}\left[X(\ell) \;\middle|\; W(\vec{r}(\ell)) \geq \frac{r' w_0}{\alpha} \right] \cdot p$$

$$+ \Pr\left[X(\ell) \;\middle|\; W(\vec{r}(\ell)) \leq \frac{r' w_0}{\alpha} \right] \cdot (1-p)$$

$$\leq \Pr_{\vec{r}}\left\{ \overline{1} \;\middle|\; W(\vec{r}(\ell)) \geq \frac{r' w_0}{\alpha} \right\} \cdot p$$

$$+ \Pr_{\vec{r}}\left\{ \overline{2} \;\middle|\; W(\vec{r}(\ell)) \leq \frac{r' w_0}{\alpha} \right\} (1-p)$$

$$\leq 1/L^2 \cdot p + \frac{1}{L^2}(1-p)$$

$$\leq \frac{1}{L^2} + \frac{1}{L^2} = \frac{2}{L^2}.$$

$\therefore \exists \ell \in [L]$ s.t $\Pr_{\vec{r}}\{X(\ell)\} \leq \frac{2}{L}$

$\therefore \Pr_{\vec{r}}\left[\forall \ell \in [L];\ \overline{X(\ell)} \right] \geq 1 - \frac{2}{L}$

$\geq 0 \qquad L > 2$

$\therefore$ exists such an r.