Lemma 4.9. *If* $|\mathcal{A}| \le (1+\xi)^{w_0}$, *then* $\operatorname{span}(\{\lambda(A) \mid A \in FPHP(G) \cup \mathcal{A}\}) \subsetneq \Lambda$.

Consider pigeon axioms P_i for $i \in [m]$.

$Z(P_i) = \{\}$ Why? → Any matching φ s.t. $i \in \operatorname{dom}(\varphi)$ will satisfy C. If $i \notin \operatorname{dom}(\varphi)$ then $\varphi \notin Z(C)$. And if $C(\varphi) = \text{True}$ then $\varphi \notin Z(C)$ again.

$\therefore \lambda(P_i) = \emptyset$

Functional axiom $F_i^{j,j'}$:

we get $Z(F_i^{j,j'}) = \{\}$ Why? As φ is a partial matching with $i \in \operatorname{dom}(\varphi)$ it cannot set φ_i to both j and j'. And any other assignment satisfies $F_i^{j,j'}$

$\therefore \lambda(F_i^{j,j'}) = \emptyset$

This leaves the hole axioms and $(w_0, \vec{d})$ - axioms

$H_j^{(i,i')}$ also has $Z(H_j^{i,i'}) = \{\}$ Why? $\Delta_G(i) - 1 \ge d_i - \delta_i$

$\Delta_G(i') - 1 \ge d_{i'} - \delta_{i'}$

↳ i and i' the only heavy pigeons

$\because$ $d_i \le \Delta_G(i)$

$\delta_i = 4\Delta_G(i)\xi \underset{\uparrow}{\ge} 2\xi\Delta > 1$

G is a $(r, \Delta, (1-2\xi)\Delta)$ boundary expander

so $\partial_G(i) = \Delta_G(i) \ge (1-2\xi)\Delta$

$\because \xi \le 1/4 \Rightarrow \Delta_G(i) \ge \frac{1}{2}\Delta$

or $4\Delta_G(i)\xi \ge 2\xi\Delta > 1$

Assuming $\Delta > 2$.

$\therefore \lambda(H_j^{i,i'}) = \emptyset$

All partial matchings that assign i and i' satisfy $H_j^{i,i'}$ because any matching cannot assign j to both i and i'.

All thats left to show is $\{\lambda(C) : C \in \mathcal{A}\}$ does not span Λ.

i.e $\dim(\downarrow) < \dim(\Lambda)$.

Lastly, $C \in \mathcal{A}$ where is $(w_0, \vec{d})$ axiom.

w/o satisfying C how many holes for i?

$$\dim(\lambda(C)) = \prod_{i \notin P_{\vec{d}}(C)} \dim(\Lambda_i) \prod_{i \in P_{\vec{d}}(C)} (\Delta_C(i) - d_i)$$

$$\le \prod_{i \notin P_{\vec{d}}(C)} \dim(\Lambda_i) \prod_{i \in P_{\vec{d}}(C)} (\Delta_G(i) - d_i)$$

→ Want to show $< \prod_{i \in P_{\vec{d}}(\mathcal{A})} \Lambda_i$

$$\frac{\dim(\lambda(c))}{\dim(\Lambda)} \leq \prod_{i \in P_d(A)} \frac{\Delta_G(i) - d_i}{\Delta_G(i) - d_i + \xi_i/4} \leq (1-\xi)^{w_0}.$$

↳ This algebra is a little unclear. See below: plot in python/ mathematica.

Now as $|A| \leq (1+\xi)^{w_0}$

$$\frac{\dim(A)}{\dim(\Lambda)} \leq (1+\xi)^{w_0}(1-\xi)^{w_0} = (1-\xi^2)^{w_0}$$

< 1 as $0 < \xi \leq 1/4$

APPENDIX

This implies $\lambda(A)$ does not span Λ;

$$\frac{\Delta_G(i) - d_i}{\Delta_G(i) - d_i + 4\xi\Delta_G(i)} \leq (1-\xi)$$

$|P_d(c)| = w_0$

$\Delta_G(i) = x \quad d_i = y \qquad \frac{x - y}{x - y + 4\varepsilon x} \leq 1 - \varepsilon$

Clearly this is < 1
but it a lot lesser
than $1-\varepsilon$; as dividing by $4\xi x$ shrinks
a lot more than subtracting ξ