PROOF: for lemma 4.10 in [Rezende' 2021] → main techical work.

Let $S = \underset{\pi,\nu}{\text{closure}}(C)$ $S_0 = \underset{\pi,\nu}{\text{closure}}(C_0)$ $S_1 = \underset{\pi,\nu}{\text{closure}}(C_1)$

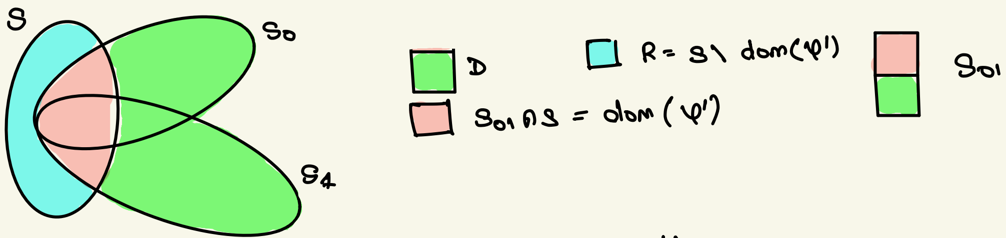
$D = S_{01} \setminus S$ where $S_{01} = S_0 \cup S_1$

Want to show $\lambda(C) \subseteq \text{span}\{\lambda(C_0), \lambda(C_1)\}$

i.e for any arbitrary $\varphi \in Z(C)$ we want to show

$$\varphi \in \text{span}\{\lambda(C_0), \lambda(C_1)\}$$

Let φ' be a restriction of φ to $S \cap S_{01}$.
ie $\text{dom}(\varphi') = S \cap S_{01}$ and $\varphi'(x) = \varphi(x)\ \forall x \in \text{dom}(\varphi')$



From the defn of λ; $\lambda(\varphi) \subseteq \lambda(\varphi')$ if φ' is a restriction of φ

so it suffices to show $\lambda(\varphi') \in \text{span}\{\lambda(C_0), \lambda(C_1)\}$

↑ This is what we will show.

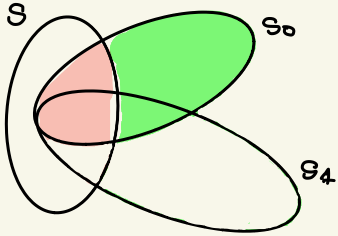
From here on let φ' will refer to the restriction of arbitrary φ

Also let $M_D = \{\hat{\varphi} : \hat{\varphi}$ created by extending $\text{dom}(\varphi')$ to D but $C(\hat{\varphi}) = \text{FALSE}\}$

Now pick arbitrary $\tilde{\varphi} \in M_D$.

Since $c(\tilde{\varphi}) = \text{False}$ $\Rightarrow$ By soundness of the resolution proof system as $c(\tilde{\varphi})$ is false it must be that either c_0 or c_1 is false.

W/o loss of generality; assume $c_0(\tilde{\varphi}) = \text{FALSE}$



Now restrict dom($\tilde{\varphi}$) to just S_0 as shown on the left $\rightarrow \tilde{\varphi}'$

As $c(\tilde{\varphi}') = \text{False}$ (restricting doesn't change truth table)

$\therefore\ \tilde{\varphi}' \in Z(c_0)$

$$\text{so}\quad \lambda(\tilde{\varphi}) \subseteq \lambda(\tilde{\varphi}') \subseteq \lambda(c_0) \subseteq \text{span}\{\lambda(c_0), \lambda(c_1)\}$$

For any $\tilde{\varphi} \in M_D$. But I wanted to show it for any φ' which is a restriction of arbitrary $\varphi \in Z(C)$

But if I show $\underline{\lambda(\varphi') \subseteq \lambda(M_D) = \text{span}\{\lambda(\tilde{\varphi}) : \tilde{\varphi} \in M_D\}}$

The thing that is left to show

it would imply

$$\boxed{\lambda(\varphi) \subseteq \lambda(\varphi') \subseteq \lambda(M_D) \subseteq \text{span}\{\lambda(c_0), \lambda(c_1)\}}$$

To show $\lambda(\varphi') \subseteq \lambda(M_D)$, it suffices to show $\bigotimes_{i\in D} \lambda_i(\tilde{\varphi}_i) = \bigotimes_{i \in D} \Omega_i$

$$\lambda(\varphi') = \bigotimes_{i \in S_0 \cap S_1} \lambda_i(\varphi'_i)\ \boxed{\bigotimes_{i\in D} \Omega_i}\ \bigotimes_{i \in R} \Omega_i$$

see why

On the other hand

$$\lambda(M_D) = \text{span}\Big\{ \bigotimes_{i \in S_0\cap S_1} \overset{\lambda_i(\tilde{\varphi}_i) \text{ as it is a restriction}}{\overset{=}{\lambda_i(\varphi'_i)}}\ \boxed{\bigotimes_{i \in D} \lambda_i(\tilde{\varphi}_i)} \bigotimes_{i\in R} \Omega_i \;\Big|\; \tilde{\varphi} \in M_D \Big\}$$

If $\boxed{}$ = $\boxed{}$, it concludes the proof

So now we are set to prove $\text{span}\{\bigotimes_{i\in D} \lambda_i(\tilde{\varphi}_i) : \tilde{\varphi} \in M_D\}$

$$= \bigotimes_{i\in D} \Lambda_i$$

Now we restate a fact about boundary expanders.

Lemma 3.4. *For G an (r, Δ, c)-boundary expander, let $T \subseteq V_{\text{exp}}(G)$ be such that $|T| \leq r$ and $|\text{closure}_{r,\nu}(T)| \leq r/2$, let $G' = G \setminus (\text{closure}_{r,\nu}(T) \cup N(\text{closure}_{r,\nu}(T)))$ and $V_{\text{exp}}(G') = V_{\text{exp}}(G) \cap V(G')$. Then any set $S \subseteq V_{\text{exp}}(G')$ of size $|S| \leq r/2$ satisfies $|\partial_{G'}(S)| \geq \nu|S|$.*

In our case $V_{exp}(G) = V_P = [m]$

$$G' = G \setminus \left(\text{closure}_{r,\nu}(C) \cup N_G(\text{closure}_{r,\nu}(C))\right) = G \setminus (S \cup N_G(S))$$

As $|D| \leq |S_0| + |S_1|$

$\leq |r/4| + |r/4|$ (by lemma assumption)

$\leq \frac{r}{2}$

$\therefore \partial_{G'}(D) \geq \underbrace{(1-3\xi)\Delta}_{\nu}|D|$ (As $|\text{closure}_{r,\nu}(C)| \leq r/4$ by assumption and invoking lemma 3.4.)

By averaging argument
$\exists j \in D$ st $\Delta_G(j) \geq (1-3\xi)\Delta$

Now all pigeons $i \in D$ are light as
$i \notin \text{closure}_{r,\nu}(C) \supseteq |P_{\vec{d},\vec{\delta}}(C)|$

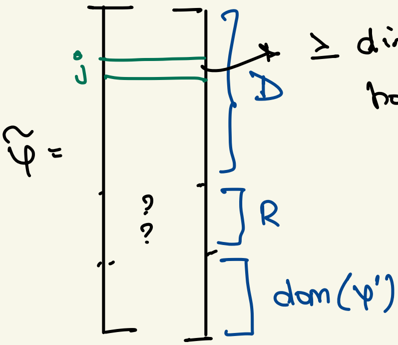
So they have at most $d_j - \delta_j$ holes according to $E(G)$ they can be mapped to. in C.
Now this means there are at least

$(1-3\xi)\Delta - (d_j - \delta_j)$ holes I can match pigeon j to w/o satisfying C.

This means M_D has at least

$$(1-3\xi)\Delta - (d_j - \delta_j) \geq (1-3\xi)\Delta_G(j) - (d_j - \overbrace{4\xi\Delta_G(j)}^{\text{def of } \delta_j})$$

$$\geq \Delta_G(j) - d_j + \frac{\delta_j}{4} = \dim(\Lambda_j)$$



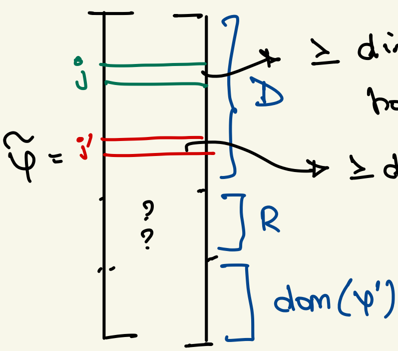
$\geq \dim(\Lambda_j)$ options for this hole w/o satisfying C.

$\therefore |M_D| \geq \dim(\Lambda_j)$

Now $D' = D \setminus \{j\} \rightarrow |D'| \leq \frac{r}{2}$ (still).

so $\partial_{G'}(D') \geq (1-3\xi)\Delta |D'|$; Once again by averaging argument $\exists\, j' \in D'$ s.t.

$$\Delta_G(j') \geq (1-3\xi)\Delta$$



$\geq \dim(\Lambda_j)$ options for this hole w/o satisfying C.

$\geq \dim(\Lambda_{j'})$ options again

$\therefore |M_D| \geq \dim(\Lambda_j) * \dim(\Lambda_{j'})$

We do this for all $i \in D \Rightarrow$ invoking the lemma $|D|$ times

→ Each time $|D'| \leq r/2$ so we are good.

$$\Rightarrow |M_D| \geq \prod_{i \in D} \dim(\Lambda_i)$$

Now as M_D is large enough for each i

$$\boxed{\lambda(M_D)} = \boxed{\bigotimes_{i \in D} \Omega_i}$$

To see why

observe from the defn of $\lambda_i : V_H \to \Omega_i$

if $J \subseteq V_H$ with $|J| \geq \dim(\Omega_i)$

then $\{\lambda_i(j) : j \in J\}$ spans Ω_i

As discussed above; for each $i \in D$; there are $\geq \dim(\Omega_i)$ holes we can map to and

This completes the proof